

Rough notes on
Binary Majority Consensus
(Episode 1: Majority protocols)

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Let $G = ([n], E)$ be a graph, let C be a finite set of *colors* and let $c : [n] \rightarrow C$ be an initial *coloring* of the nodes of G . If the number of colors is $|C| = h$ we will call c an h -*coloring*. Consider the following family of protocols

Protocol 1 k -majority

At every step each node independently picks k neighbors (including itself and with repetition) u.a.r. and recolors itself according to the majority of the colors it sees.

Our first main goal for this line of research on *distributed community detection and beyond* would be to prove the following conjecture.

Conjecture 1 *Let $G \equiv K_n$ be the clique with n -nodes. For every initial h -coloring with $2 \leq h \leq \log n$, if each node runs the k -majority protocol with $2 \leq k \leq n$, then after $\mathcal{O}(\log n)$ time steps all nodes have the same color w.h.p.*

1 Unbalanced 2-coloring with 3-majority

We start analyzing the 3-majority protocol in the case of 2-colorings. For a 2-coloring $c : [n] \rightarrow \{\mathbf{red}, \mathbf{blue}\}$, we say that c is ω -unbalanced, if the difference between the number of **red** and **blue** nodes in absolute value is at least ω . In the next lemma we show that, if the initial configuration is sufficiently unbalanced, then the conjecture is true.

Lemma 2 *If $G \equiv K_n$, the initial 2-coloring is $\Omega(\sqrt{n \log n})$ -unbalanced, and each node runs the 3-majority protocol, then after $\mathcal{O}(\log n)$ time steps all nodes have the same color w.h.p.*

Proof. Let X_t be the random variable counting the number of **red** nodes at time t . For every node i let Y_i the indicator random variable of the event “node i is **red** at the next step”. For every $a = 0, 1, \dots, n$ it holds that

$$\mathbf{P}(Y_i = 1 \mid X_t = a) = \left(\frac{a}{n}\right)^3 + 3\frac{a^2(n-a)}{n^3} = \frac{a^2}{n^3}(3n - 2a)$$

Hence, the expected number of **red** nodes at the next time step is

$$\mathbf{E}[X_{t+1} \mid X_t = a] = \left(\frac{a}{n}\right)^2 (3n - 2a) \tag{1}$$

We split the analysis in three phases.

Phase 1: From $n/2 - \Theta(\sqrt{n \log n})$ to $n/4$:

Suppose that the number of **red** nodes is $X_t = a$ for some $a \leq n/2 - s$ where $c\sqrt{n \log n} \leq s \leq n/4$ for some positive constant c . Now we show that $X_{t+1} \leq n/2 - (9/8)s$ w.h.p.

Observe that function $f(a) = a^2(3n - 2a)$ in (??) is increasing for every $0 < a < n$. Hence, for $a \leq n/2 - s$ we have that

$$\begin{aligned} \mathbf{E}[X_{t+1} | X_t = a] &= \left(\frac{a}{n}\right)^2 (3n - 2a) \leq \left(\frac{n}{2} - s\right)^2 (3n - 2(n/2 - s)) \\ &= \frac{n}{2} - \frac{3}{2} \cdot s + 2 \cdot \frac{s^3}{n^2} \\ &\leq \frac{n}{2} - \frac{3}{2} \cdot s + s \cdot \left(2 \cdot \frac{s^2}{n^2}\right) \\ \text{since } s \leq n/4 &\leq \frac{n}{2} - \frac{5}{4} \cdot s. \end{aligned}$$

Notice that random variables Y_i 's are independent conditional on X_t . From Chernoff bound (Lemma ??) it thus follows that, for every $a \leq s \leq n/4$ it holds that

$$\mathbf{P}\left(X_{t+1} \geq \frac{n}{2} - \frac{9}{8} \cdot s \mid X_t = a\right) \leq e^{-s^2/(64n)} \quad (2)$$

If $s \geq c\sqrt{n \log n}$ we have that $X_{t+1} \leq (n/2) - (9/8)s$ w.h.p. Thus, when $c\sqrt{n \log n} \leq s \leq n/4$ the *unbalance* of the coloring increases exponentially w.h.p.

Let us name \mathcal{E}_t the event

$$\mathcal{E}_t = "X_t \leq \max\left\{\frac{n}{4}, \frac{n}{2} - (9/8)^t\right\}"$$

Observe that from (??) it follows that, for every $t \in \mathbb{N}$, we have

$$\mathbf{P}\left(\mathcal{E}_{t+1} \mid \bigcap_{i=1}^t \mathcal{E}_i\right) \geq 1 - n^{-\alpha}$$

Thus, for $T = \frac{\log(n/4)}{\log(9/8)} = \mathcal{O}(\log n)$ the probability that the number of **red** nodes has gone below $n/4$ within the first T time steps is

$$\begin{aligned} \mathbf{P}(\exists t \in [0, T] : X_t \leq n/4) &\geq \mathbf{P}\left(\bigcap_{t=1}^T \mathcal{E}_t\right) \geq \prod_{t=1}^T \mathbf{P}\left(\mathcal{E}_t \mid \bigcap_{i=1}^{t-1} \mathcal{E}_i\right) \\ &\geq (1 - n^{-\alpha})^T \geq 1 - 2Tn^{-\alpha} \geq 1 - n^{-\alpha/2} \end{aligned}$$

Phase 2: From $n/4$ to $\mathcal{O}(\log n)$: If $X_t = a$ with $a \leq (1/4)n$, from (??) it follows that

$$\mathbf{E}[X_{t+1} | X_t = a] = \left(\frac{a}{n}\right)^2 (3n - 2a) \leq \frac{3}{4}a$$

and from Chernoff bound (Lemma ??) it follows that

$$\mathbf{P}\left(X_{t+1} \geq \frac{4}{5}a \mid X_t = a\right) \leq e^{-\beta a}$$

for a suitable positive constant β . Hence if $a = \Omega(\log n)$ then the number of **red** nodes decreases exponentially w.h.p. By reasoning as in the previous phase we get that after further $\mathcal{O}(\log n)$ time steps the number of **red** nodes is $\mathcal{O}(\log n)$.

Phase 3: From $\mathcal{O}(\log n)$ to zero: Observe that for $a = \mathcal{O}(\log n)$, in (??) we have that

$$\mathbf{E}[X_{t+1} | X_t = a] \leq c/n$$

for a suitable positive constant c . Hence, by using Markov inequality $\mathbf{P}(X_{t+1} \geq 1 | X_t = a) \leq c/n$ and since X_{t+1} is integer valued it follows that all nodes are **blue** w.h.p. \square

In the previous lemma we showed that, if the 3-majority protocol starts from a 2-coloring that is sufficiently unbalanced then after $\mathcal{O}(\log n)$ time steps the graph is monochromatic. A natural question is whether the lemma still holds if we use the 2-majority protocol.

For the 2-majority protocol over a 2-coloring we need to specify a way of breaking ties. A natural way for that is the *inertial* way: *In case of ties keep your current color.*

Exercise 3 Show that if each node runs the 2-majority protocol with inertia then, if we start from a $\Theta(\sqrt{n \log n})$ -unbalanced 2-coloring, after $\mathcal{O}(\log n)$ time steps all nodes have the same color w.h.p.

2 Conclusions

In this episode we proved that if we start from a sufficiently unbalanced 2-coloring on a clique and we run the 3-majority protocol we will end-up in a monochromatic graph in $\mathcal{O}(\log n)$ time steps. This result easily extends to the 2-majority protocol. What happens if the initial configuration is not unbalanced? and if we have more colors?

See you on the next episode...

Appendix

Lemma 4 (Chernoff bound, additive form) Let $X = \sum_{i=1}^n X_i$ where X_i 's are independent Bernoulli random variables and let $\mu = \mathbf{E}[X]$. Then for every $\lambda > 0$ it holds that

$$\mathbf{P}(X \geq \mu + \lambda) \leq e^{-2\lambda^2/n}$$

Lemma 5 (Chernoff bound, multiplicative form) Let $X = \sum_{i=1}^n X_i$ where X_i 's are independent Bernoulli random variables and let $\lambda \geq \mathbf{E}[X]$. Then for $0 < \delta < 1$ it holds that

$$\mathbf{P}(X \geq (1 + \delta)\lambda) \leq e^{-(\delta^2/3)\lambda}$$